QUALITATIVE ANALYSIS OF FREE VIBRATIONS OF AN ELASTIC THIN SHELL

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An account is given of an asymptotic method of integrating the dynamic equations of the classical linear theory of thin elastic shells for the problem of free vibrations. It represents a dynamic analogy of the asymptotic method developed for the static problem [1].

The present method is used to analyze the asymptotic properties of the frequencies, the associated states of stress, and their dependence on the order of smallness of the dimensionless thickness of the shell, and on the density and configuration of the nodal lines. A classification is made of the forms of the free vibrations and, for each of these, simplified equations are derived for their determination in first-order approximation. A qualitative analysis is also made of the spectrum of the eigenfrequencies of the shell.

Methods of integration of the obtained approximate equations are not considered. Discussion is limited to particular features of the corresponding boundary-value problems (if they have not been discussed in the problems under consideration).

1. In our study of the free vibrations of a thin elastic shell the equations and formulas of the theory of momentums will be used at a starting point.

Equations of equilibrium are

$$\frac{1}{A}\frac{\partial T_1}{\partial \alpha}+\frac{1}{AB}\frac{\partial B}{\partial \alpha}(T_1-T_2)+\frac{1}{B}\frac{\partial S}{\partial \beta}+\frac{2}{AB}\frac{\partial A}{\partial \beta}S-\left(\frac{N_1}{R_1'}-\frac{N_2}{R_{12}}\right)+\lambda\xi=0 \quad (\alpha\beta)$$

$$\frac{T_1}{R_1'}+\frac{T_2}{R_2'}-\frac{2S}{R_{12}}+\frac{1}{AB}\left[\frac{\partial}{\partial \alpha}(BN_1)+\frac{\partial}{\partial \beta}(AN_2)\right]+\lambda\zeta=0 \qquad (1.1)$$

$$\frac{1}{A}\frac{\partial G_1}{\partial \alpha}+\frac{1}{AB}\frac{\partial B}{\partial \alpha}(G_1-G_2)-\frac{1}{B}\frac{\partial H}{\partial \beta}-\frac{2}{AB}\frac{\partial A}{\partial \beta}H-N_1=0 \quad (\alpha\beta) \qquad (1.2)$$

and the elasticity relations are given by

$$T_1 = \frac{2Eh}{1-\sigma^2}(\varepsilon_1 + \sigma\varepsilon_2) \quad (\alpha\beta), \qquad S = \frac{Eh}{1+\sigma}\omega \tag{1.3}$$

$$G_1 = -\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_1 + \sigma\varkappa_2) \quad (\alpha\beta), \qquad H = \frac{2Eh^3}{3(1+\sigma)}\tau \tag{1.4}$$

while

$$2Eh\varepsilon_1 = \frac{1}{A}\frac{\partial\xi}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\eta - \frac{\zeta}{R_1'} \quad (\alpha\beta)$$

$$2Eh\omega = \frac{A}{B}\frac{\partial}{\partial\beta}\frac{\xi}{A} + \frac{B}{A}\frac{\partial}{\partial\alpha}\frac{\eta}{B} + \frac{2\zeta}{R_{12}} \tag{1.5}$$

$$2Eh\varkappa_1 = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial\zeta}{\partial\alpha} + \frac{\xi}{R_1'} - \frac{\eta}{R_{12}}\right) + \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{B}\frac{\partial\zeta}{\partial\beta} + \frac{\eta}{R_2'} - \frac{\xi}{R_{12}}\right) + $$
$$+ \frac{1}{2}\frac{1}{R_{12}}\frac{1}{AB}\left[\frac{\partial}{\partial\beta}(A\xi) - \frac{\partial}{\partial\alpha}(B\eta)\right] \quad (\alpha\beta) \tag{1.6}$$

$$2Eh\tau = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{B}\frac{\partial\zeta}{\partial\beta} + \frac{\eta}{R_2'} - \frac{\xi}{R_{12}}\right) - \frac{1}{AB}\frac{\partial A}{\partial\beta}\left(\frac{1}{A}\frac{\partial\zeta}{\partial\alpha} + \frac{\xi}{R_1'} - \frac{\eta}{R_{12}}\right) + $$
$$+ \frac{1}{R_1'}\left(\frac{1}{B}\frac{\partial\xi}{\partial\beta} - \frac{1}{AB}\frac{\partial B}{\partial\alpha}\eta + \frac{\zeta}{R_{12}}\right) - \frac{1}{R_{12}}\left(\frac{1}{B}\frac{\partial\eta}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}\xi - \frac{\zeta}{R_2'}\right)$$

are the strain-displacement equations. The notation of [1] is used and α and β are the dimensionless parameters of an arbitrary system of orthogonal coordinates. The symbol $(\alpha\beta)$ means that the equation preceding it can be transformed into another equation by substituting α, 1, A, and ξ with β, 2, B, and η, respectively. The parameter λ and the quantities ξ, η, and ζ are defined by the formulas

$$\lambda = \frac{m\omega^2}{2Eh}, \qquad \xi = 2Ehu, \qquad \eta = 2Ehv, \qquad \zeta = 2Ehw$$

where m is the mass per unit median surface area of the shell, and ω is the frequency of vibration. (For simplicity, ξ, η, ζ will henceforth by called displacements.)

We shall assume that the shell executes harmonic oscillations and that the factor $\sin\omega t$ appearing in the unknown magnitudes, will be neglected.

We shall also assume that the shell has two closed boundaries which (after a suitable transformation of the orthogonal curvilinear coordinates, if necessary) coincide with the lines $\alpha = \alpha_1$, and $\alpha = \alpha_2$. One of these boundaries can degenerate into a point, in which case its boundary conditions must be replaced by the conditions of continuity.

2. According to how the β-curves, i.e. the family of curves containing the boundary of the shell, are distributed, we shall distinguish the following cases:

case I*a* $\quad 1/R_2' \neq 0, \quad 1/R_{12} \neq 0$
case I*b* $\quad 1/R_2' \neq 0, \quad 1/R_{12} = 0$

(the β-curves do not follow the asymptotic lines. In the case I*a* they do not coincide with the lines of curvature, while in the case I*b*, they do.)

case II*a* $1 / R_2' = 0, \quad 1 / R_{12} \neq 0, \quad 1 / R_1' \neq 0$
case II*b* $1 / R_2' = 0, \quad 1 / R_{12} \neq 0, \quad 1 / R_1' = 0$

(Here, the β-curves follow one of the two families of asymptotic lines. In the case II*a* they are not orthogonal to the other family, while and in the case II*b*, they are.)

case III $1 / R_2' = 0, \quad 1 / R_{12} = 0, \quad 1 / R_1' \neq 0$

(In this case, the β-curves follow a unique family of asymptotic curves.)

Case I can occur with shells of arbitrary curvature, case II is possible only when the shell has negative curvature (II*a* for non-minimal surfaces and case II*b* for the minimal ones), and case III can occur only when the shell has zero curvature.

The calculations will be carried out for all the cases simultaneously. When necessary, the respective formulas will be distinguished by the corresponding Roman numerals with or without a letter.

3. In the present paper main consideration is devoted to vibrations with sufficiently large variations of the states of stress and strain, where the term variation is used in the same sense as in the static theory of shells [1]. The solution obtained is very approximate (the possibility of improving the accuracy is discussed in section 9.) Accordingly, when studying different types of integrals of the dynamic equations of the theory of shells in sections 4 to 8, we have in each equation retained only the principal (for the given integral) terms. In estimating the different terms, we assume that the differentiation symbols in front of the unknowns (displacements, forces, and moments) obey the relations

$$\frac{\partial}{\partial \alpha} \sim h_*^{-p}, \qquad \frac{\partial}{\partial \beta} \sim h_*^{-q} \qquad \left(h_* = \frac{h}{R}\right) \tag{3.1}$$

where $h_* = h / R$ is the dimensionless half-thickness of the shell, R is a characteristic radius of curvature of the median surface, and p, and q are the indices of variation in the directions of the α- and β-curves, respectively.

We shall assume everywhere that

$$0 < \max(p, q) < 1 \tag{3.2}$$

Here, the left-hand side inequality guarantees the applicability of the method of investigation used here, and the right-hand side inequality guarantees the applicability of equations (1.1) to (1.6).

We shall also assume that

$$\lambda = \frac{m\omega^2}{2Eh} \sim h_*^{2r} \tag{3.3}$$

where r is a number characterizing the asymptotic magnitude of the frequency of oscillation (ω decreases like h_*^{2r} as r increases).

4. The basic integrals for quasi-transverse oscillations will be defined as the solutions of the dynamic equations of the theory of shells for which the following asymptotic relation is satisfied

$$\max(\xi, \eta) << \zeta \tag{4.1}$$

and the displacements ξ, η, and ζ are determined in the first approximation by the equations

$$\begin{gathered}
\frac{1}{A}\frac{\partial T_1}{\partial\alpha} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}(T_1 - T_2) + \frac{1}{B}\frac{\partial S}{\partial\beta} + \frac{2}{AB}\frac{\partial A}{\partial\beta}S = 0 \quad (\alpha\beta) \\
\frac{T_1}{R_1'} + \frac{T_2}{R_2'} - \frac{2S}{R_{12}} + \lambda\zeta = 0 \\
T_1 = \frac{2Eh}{1-\sigma^2}(\varepsilon_1 + \sigma\varepsilon_2) \quad (\alpha\beta), \qquad S = \frac{Eh}{1+\sigma}\omega \\
2Eh\varepsilon_1 = \frac{1}{A}\frac{\partial\xi}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\eta - \frac{\zeta}{R_1'} \quad (\alpha\beta) \\
2Eh\omega = \frac{A}{B}\frac{\partial}{\partial\beta}\frac{\xi}{A} + \frac{B}{A}\frac{\partial}{\partial\alpha}\frac{\eta}{B} + \frac{2\zeta}{R_{12}}
\end{gathered} \tag{4.2}$$

These are the dynamic equations of the momentumless theory in which the inertial membrane forces are disregarded.

Thus, the basic integrals for quasi-transverse oscillations represent the dynamic analogy of these solutions of the static equations of the theory of shells which we called the basic integrals [1]. The analogy is also evident in the fact that, when (4.2) is integrated,only two boundary conditions can be satisfied on the boundary.

The asymptotic properties of the basic integrals for the quasi-transverse oscillations are given in the Table.

	Ia	*Ib*	*IIa*	*IIb*	III
ξ	p	p	p	$2p-q$	p
η	p	$2p-q$	p	p	$2p-q$
T_1	$\frac{2p-2q}{p}$	$\frac{2p-2q}{p}$	$\frac{3p-3q}{2p-q}$	$\frac{3p-3q}{2p-q}$	$4p-4q$
T_2	0	0	$p-q$	$p-q$	$2p-2q$
S	$p-q$	$p-q$	$2p-2q$	$2p-2q$	$3p-3q$
$2Eh\varepsilon_1$	0	0	$p-q$	$p-q$	$2p-2q$
$2Eh\varepsilon_2$	0	0	$p-q$	$p-q$	$2p-2q$
$2Eh\omega$	$p-q$	$p-q$	$2p-2q$	$2p-2q$	$3p-3q$
λ	0	0	$2p-2q$	$2p-2q$	$4p-4q$
Region of formal existence	$p<1/2$, $q<1/2$, $3q-p<1$	$p<1/2$, $q<1/2$, $4q-2p<1$	$3p-q<1$, $q<1$	$3p-q<1$, $3q-p<1$	$4p-2q<1$, $q<1/2$

The Table gives the exponents s in the asymptotic relation $A \sim h_*^s$, where A denotes the quantity indicated in the first column. In the case of force T_1, one must choose the smaller of the two values of s separated by the division line. A freedom of choice of the scale factor for the considered homogeneous problem has been utilised, and is assumed (here and in the following), that $\zeta \sim h_*{}^0$. The formulas (3.1) were taken into account, and for definiteness it is assumed that $p \geqslant q$. As a check of the last column in the table it is necessary to recall, that in case III $\partial B/\partial\alpha = 0$.

The Table and the formulas (3.1) make it possible to obtain asymptotic estimates of all the terms in equations (4.2) and to determine the principal (commensurable with the lowest degree of h_*) terms in each of them. Neglecting the remaining terms, one can construct a system for the determination of the above-mentioned integrals in the same approximation. This system must contain no obvious discrepancies and in fact, it should contain as many unknowns as there are equations, while in any subsystem belonging to it, the number of equations should not exceed the number of unknowns. The choice of the asymptotic properties of the basic integrals for the quasi-transverse oscillations quoted in the Table is, in fact, based on this requirement.

It is not difficult to establish also the asymptotic properties of the moments and shearing forces. With the aid of (1.6), (1.4), and (1.1), these quantities can be expressed in terms of ξ, η, and ζ by formulas that contain only linear action terms. From these it follows that in all cases (see section 2)

$$G_1 \sim G_2 \sim h_*^{2-2p}; \qquad H \sim h_*^{2-p-q}; \qquad N_1 \sim h_*^{2-3p}, \qquad N_2 \sim h_*^{2-2p-q} \tag{4.3}$$

By making use of the Table and relations (3.1) and (4.3), one can estimate all the terms of the original dynamic equations (1.1) to (1.6) and determine the inequalities which must be satisfied by p and q in order to ensure that the quantities neglected in transition from (1.1) to (1.4) are smaller than the retained ones, when h_* is sufficiently small. This gives the left-hand side inequality (3.2) and two additional inequalities given in the last row of the Table. (In this row the assumption that $p \geqslant q$ has been rejected.) For cases Ia and Ib, two versions of the second inequality are possible (they are separated by a horizontal line). The upper one is valid when $1/R_1' \neq 0$, and the lower one when $1/R_1' = 0$. In the following, we will mean that such inequalities determine the region of formal existence of the integral of the given type.

5. Basic integrals for quasi-membrane oscillations denote solutions of the dynamic equations of the theory of shells for which the asymptotic relation

$$\max(\xi, \eta) \gg \zeta \tag{5.1}$$

holds, and for which the membrane displacements ξ and η and the membrane forces T_1, T_2, and S are determined by the equations

$$\frac{1}{A}\frac{\partial T_1}{\partial\alpha} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}(T_1 - T_2) + \frac{1}{B}\frac{\partial S}{\partial\beta} + \frac{2}{AB}\frac{\partial A}{\partial\beta}S + \lambda\xi = 0 \qquad (\alpha\beta)$$
$$T_1 = \frac{1}{1-\sigma^2}\left(\frac{1}{A}\frac{\partial\xi}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}\eta + \frac{\sigma}{B}\frac{\partial\eta}{\partial\beta} + \frac{\sigma}{AB}\frac{\partial B}{\partial\alpha}\xi\right) \quad (\alpha\beta) \tag{5.2}$$
$$S = \frac{1}{2(1+\sigma)}\left(\frac{B}{A}\frac{\partial}{\partial\alpha}\frac{\eta}{B} + \frac{A}{B}\frac{\partial}{\partial\beta}\frac{\xi}{A}\right)$$

The normal displacement, the moments, and the shearing forces can be expressed in terms of unknowns present in the system (5.2) by means of linear relations: ζ is determined from the equation

$$\frac{T_1}{R_1'} + \frac{T_2}{R_2'} - \frac{2S}{R_{12}} + \lambda\zeta = 0 \tag{5.3}$$

and G_1, G_2, H, N_1, and N_2, are constructed in the way described in section 4.

The asymptotic properties of the basic integrals for quasi-membrane oscillations when $p = q$, i.e. for the same index of variation in the directions of the α- and β-curves, can be expressed by the relations

$$\begin{aligned} &\xi \sim \eta \sim h_*^{-p}, \qquad \zeta \sim h_*^{\circ}, \qquad T_1 \sim T_2 \sim S \sim h_*^{-2p} \\ &G_1 \sim G_2 \sim H \sim h_*^{2-2p}, \quad N_1 \sim N_2 \sim h_*^{3-3p}, \qquad \lambda \sim h_*^{-2p} \end{aligned} \tag{5.4}$$

With the aid of (5.4) and (3.1), it is easy to derive the inequalities for p and q, which determine the region of formal existence (see section 4) of the basic integrals for quasi-membrane oscillations. These inequalities coincide exactly with (3.2), i.e. they are satisfied whenever both, the present method of investigation and the classical theory of shells are simultaneously applicable.

The basic integrals for quasi-transverse oscillations do not have an analogy in the asymptotic theory of solution of the static problem. Equations (5.2) become the equations of the plane problem in the theory of elasticity (in the distorted metric). When integrating these equations, we can also take into account just two boundary conditions at every point of this boundary.

6. The integrals for quasi-transverse vibrations with large variations are defined as the solutions of the equations of the dynamic theory of shells, in which the normal displacement, the moments, and the shear forces are determined by the equations

$$\begin{aligned} &\frac{1}{A}\frac{\partial N_1}{\partial \alpha} + \frac{1}{B}\frac{\partial N_2}{\partial \beta} + \lambda\zeta = 0, \qquad G_1 = -\frac{2Eh^3}{3(1-\sigma^2)}(\varkappa_1 + \sigma\varkappa_2) \quad (\alpha\beta) \\ &2Eh\varkappa_1 = \frac{1}{A^2}\frac{\partial^2\zeta}{\partial\alpha^2} \quad (\alpha\beta) \quad 2Eh\tau = \frac{1}{AB}\frac{\partial^2\zeta}{\partial\alpha\partial\beta}, \quad H = \frac{2Eh^3}{3(1+\sigma)}\tau \end{aligned} \tag{6.1}$$

i.e. by the equations of the flexure theory of vibrating plates. (In section 6, we investigate vibrations with large variations, hence A and B are assumed to be constants.)

In the integrals for quasi-transverse vibrations with large variations, the asymptotic properties of the state of stress are the same as those in the case of the basic integrals for quasi-transverse vibrations (see section 4), and for parameter λ in all cases (section 2) we have the relation

$$\lambda \sim h_*^{2-4\gamma}, \qquad \gamma = \max(p, q) \tag{6.2}$$

The tangential displacements ξ and η for the present integrals can be determined in first approximation by the equations

$$\frac{1}{A}\frac{\partial T_1}{\partial \alpha}+\frac{1}{B}\frac{\partial S}{\partial \beta}=0 \qquad (\alpha\beta)$$

$$T_1=\frac{1}{1-\sigma^2}\left[\frac{1}{A}\frac{\partial \xi}{\partial a}-\frac{\zeta}{R_1'}+\nu\left(\frac{1}{B}\frac{\partial \eta}{\partial \beta}-\frac{\zeta}{R_2'}\right)\right] \qquad (\alpha\beta) \tag{6.3}$$

$$S=\frac{1}{2(1+\sigma)}\left(\frac{1}{A}\frac{\partial \eta}{\partial \alpha}+\frac{1}{B}\frac{\partial \xi}{\partial \beta}+\frac{2\zeta}{R_{12}}\right)$$

In the above equations the quantity ζ must be regarded as known, hence (6.3) represent the nonhomogeneous equations of the plane problem of the theory of elasticity.

The integrals for quasi-transverse oscillations with large variations can exist only, when at least once of the inequalities in the last row of the table is invalid. This means that their region of formal existence is defined by

$$\max(p,\ q)>{}^1/_2 \qquad (I) \quad \text{when} \quad 1/R_1'\neq 0 \tag{6.4}$$

$$\max(p,\ {}^3/_2q-{}^1/_2p)>{}^1/_2 \qquad (\mathrm{I}a)$$

$$\max(p,\ 2q-p)>{}^1/_2 \qquad (\mathrm{I}b) \qquad \text{when} \quad 1/R_1'=0$$

$$\max({}^3/_2p-{}^1/_2q,\ q)>{}^1/_2 \quad (\mathrm{II}a), \qquad \max({}^3/_2p-{}^1/_2q,\ {}^3/_2q-{}^1/_2p)>{}^1/_2 \quad (\mathrm{II}b)$$

$$\max(2p-q;\ q)>{}^1/_2 \qquad (\mathrm{III})$$

The static analogy of the above integrals, will be the integrals with large variations determined by the states of flexural and membrane stresses (cf. [1], part IV, Section 15). The approximate equations (6.1) and (6.3) as a whole, are sufficiently arbitrary to satisfy all the four boundary conditions at all points of the edge of the shell.

7. The basic integrals introduced in sections 4 and 5 are not sufficiently arbitrary to satisfy all the boundary conditions of the theory of shells and, generally speaking, in solving the boundary-value problems with the aid of the corresponding approximate equations discrepancies in the boundary conditions will arise. In connection with that, we shall introduce auxiliary integrals which will enable us to remove such discrepancies.

In the present work, it is assumed that the boundary conditions are imposed along the curves α = const and, consequently, that the discrepancies should also be removed along these curves. Taking these requirements into account, we will formulate the properties that determine the auxiliary integral: if p' and q are the indices of variation in the directions of the α- and β-curves, then the following inequality should be satisfied

$$p'>q\geqslant 0 \tag{7.1}$$

and, moreover, the normal displacement ζ should in the first approximation be determined by the equations

$$\frac{T_2}{R_2'}+\frac{1}{A}\frac{\partial N_1}{\partial \alpha}+\lambda\zeta=0, \qquad T_2=\frac{2Eh}{1-\sigma^2}(\varepsilon_2+\sigma\varepsilon_1), \qquad N_1=\frac{1}{A}\frac{\partial G_1}{\partial \alpha} \tag{7.2}$$

$$2Eh\varepsilon_2=-\frac{\zeta}{R_2'} \quad (I), \qquad 2Eh\varepsilon_r=\frac{1}{B}\frac{\partial \eta}{\partial \beta} \quad (\mathrm{II}),(\mathrm{III}), \qquad 2Eh\varkappa_1=\frac{1}{A^2}\frac{\partial^2 \zeta}{\partial \alpha^2}$$

$$G_1 = -\frac{2Eh^3}{3(1-\sigma^2)}\varkappa_1, \qquad \varepsilon_1 + \sigma\varepsilon_2 = 0 \tag{7.3}$$

which represent the dynamic analogy of the approximate equations in the theory of the simple boundary effect [1].

Eliminating all unknowns except ζ from (7.2), we obtain the equation

$$\frac{h^2}{3(1-\sigma^2)}\frac{1}{A^4}\frac{\partial^4\zeta}{\partial\alpha^4} + \left(\frac{1}{R_2'^2} - \lambda\right)\zeta = 0 \tag{7.4}$$

which differs from the governing equation in the static theory of the simple boundary effect only by the presence of the dynamic term $\lambda\zeta$.

In deriving (7.4), account has been taken of the fact that the index of variation with respect to α is certainly positive, and therefore the quantities A, B, R_1', R_2' and R_{12} in the first approximation equations can be regarded as constants (with respect to α). Within the limits of this accuracy, it is also possible to express the remaining unknowns in terms of ζ

$$T_1 = \frac{1}{B^2}\frac{\partial^2 c}{\partial\beta^2} \quad (p' < 2q), \qquad T_1 = \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial c}{\partial\alpha} \quad (p' > 2q)$$

$$T_2 = \frac{1}{A^2}\frac{\partial^2 c}{\partial\alpha^2}, \qquad S = -\frac{1}{AB}\frac{\partial^2 c}{\partial\alpha\partial\beta}$$

$$\xi = -\frac{h^2}{3(1-\sigma^2)(1/R_2'^2 - \lambda)}\left(\frac{\nu}{R_2'} + \frac{1}{R_1'}\right)\frac{1}{A^3}\frac{\partial^3\zeta}{\partial\alpha^3} \qquad (\text{I, II}a\text{, III})$$

$$\xi = \frac{h^2}{3(1-\sigma^2)\lambda}\frac{2\nu}{R_{12}}\frac{1}{A^2B}\frac{\partial^3\zeta}{\partial\alpha^2\partial\beta} \qquad (\text{II}b)$$

$$\eta = -\frac{h^2}{3(1-\sigma^2)(1/R_2'^2 - \lambda)}\frac{2}{R_{12}}\frac{1}{A^3}\frac{\partial^3\zeta}{\partial\alpha^3} \qquad (\text{I, II}) \tag{7.5}$$

$$\eta = -\frac{h^2}{3(1-\sigma^2)\lambda}\frac{1}{R_1'}\frac{1}{A^2B}\frac{\partial^3\zeta}{\partial\alpha^2\partial\beta} \qquad (\text{III})$$

$$c = \frac{h^2}{3(1-\sigma^2)(1/R_2'^2 - \lambda)}\frac{1}{R_2'}\frac{1}{A^2}\frac{\partial^2\zeta}{\partial\alpha^2} \quad (I), \qquad c = -\frac{h^2}{3(1-\sigma^2)\lambda}\frac{2}{R_{12}}\frac{1}{AB}\frac{\partial^2\zeta}{\partial\alpha\,\partial\beta} \quad (\text{II})$$

$$c = -\frac{h^2}{3(1-\sigma^2)\lambda}\frac{1}{R_1'}\frac{1}{B^2}\frac{\partial^2\zeta}{\partial\beta^2} \qquad (\text{III})$$

These formulas are derived in the same way as those in the static theory of the simple boundary effect. They are valid, provided that the inequality (7.1) is satisfied and $1/R_2'^2$ is not too small.

8. Let us assume that the given auxillary integral corresponds to the given basic integral when their indices of variation q coincide and the parameters λ have the same values.

When the auxiliary integral corresponds to the integral of the quasi-transverse vibrations, it will be called the auxiliary integral of the quasi-transverse vibrations. In this case, λ is commensurable with the exponents of h_* which are given in the Table.

On account of this, one can derive from equation (7.4) similar formulas for the index of variation p':

$$\text{when } p \geqslant q \qquad (8.0)$$
$$p' = {}^1/_2 \quad (I); \quad p' = {}^1/_2 - {}^1/_2\,(p - q) \quad (II), \quad p' = {}^1/_2 - p + q \quad (III)$$
$$\text{when } q \geqslant p$$
$$p' = {}^1/_2 - {}^1/_2\,(q - p) \quad (IIb), \quad p' = {}^1/_2 \quad (I, IIa, III)$$

It is easily verified that, whenever the inequalities in the last line of the Table are satisfied, the inequalities (7.1) are also satisfied. This means that the region of formal existence of the basic integral of quasi-transverse oscillations and the corresponding auxiliary integral are identical.

If the auxiliary integral corresponds to the basic integral of the quasi-membrane vibrations, then it will be called the auxiliary integral of the quasi-membrane vibrations. In this case we have for λ, the asymptotic estimate (5.4) which, in the more general case $(p \neq q)$ assumes the form

$$\lambda \sim h_*^{-2\gamma}, \qquad \gamma = \max\,(p,\,q) \tag{8.1}$$

From which, with the help of (7.4), we obtain

$$p' = {}^1/_2 + {}^1/_2\gamma \tag{8.1a}$$

This means that the left-hand side of relation (7.1) will be satisfied identically. Consequently, for an arbitrary basic integral of the quasi-membrane vibrations one can construct a corresponding auxiliary integral, provided the boundary forming its base does not have points where $\lambda - 1\,/\,R_2'$ is small.

The auxiliary integral of the quasi-transverse vibrations can be regarded as the dynamic analog of the simple boundary effect in the static problem. However, they also differ in one fundamental point. The simple boundary effect breaks down at the points where a pure geometrical equality $R_2' = \infty$, is fulfilled and, if it exists, then its oscillations are always damped. The auxiliary integral for the quasi-transverse vibrations degenerates at the points the properties of which depend on the form of the considered oscillations, and, if this integral exists, it can have a purely oscillatory part (when $\lambda > 1\,/\,R'^2_2$).

9. For each of the forms of integrals introduce in sections 4 to 8, approximate governing equations and formulas have been derived. This means that if the original dynamic equations of the theory of shells is written in the form

$$L\,(R) = 0 \tag{9.1}$$

where the symbols L and R stand for a differential operator and the totality of unknown variables in the theory of shells, respectively, it can be asserted that the indicated methods (different for different types of integrals) yield Equation (9.1) in the form

$$L(R) \equiv L'(R) + l(R) = 0 \tag{9.2}$$

where L' amd l are the principal and minor parts of the operator L, and where we have assumed that R can be determined approximately from the

$$L'(R) = 0$$

Asymptotic estimates of the quantities $L'(R)$ and $l(R)$ are given and for p and q we have derived such inequalities that when they are fulfilled at sufficiently small h_*, then

$$|L'(R)| \gg |l(R)| \tag{9.3}$$

is valid. This means that, in each equation separately, the moduli of the terms in $L'(R)$ significantly exceed the moduli of those in $l(R)$.

The dynamic equations of the theory of shells (9.1) can be solved by means of the method of successive approximations, putting

$$R = R_0 + R_1 + \ldots + R_s$$

and assuming that R_t satisfy the equations

$$L'(R_0) = 0, \quad L'(R_t) = -l(R_{t-1}) \quad (0 < t < s), \quad L(R_s) = -l(R_{s-1})$$

In order that the process should have asymptotic character, it is necessary (but, of course insufficient) that inequality (9.3) be satisfied. In this connection, the region of values of p and q in which (9.3) is satisfied was called the region of formal existence of the given integral.

A better-founded study of the conditions of existence of the integrals introduced in sections 4 to 8 would lead to the formulation of certain auxiliary requirements similar to those brought to light during the asymptotic analysis of the static problem (for example, the requirement that the cylindrical shell should not be too long or that the conical shell should not contain the apex, etc.).

10. In what follows, without explicitly mentioning it, we will limit ourselves to cases where the boundary conditions can be separated into membrane (tangential) and non-membrane (non-tangential) parts (cf. [1], part II, section 3). Then, by analogy with static problems, it is natural to assume that in dynamics also one can apply the method of decomposing the state of stress. The first variant of this method consi sts in the fact that the solution in the first approximation is sought in the form of a sum of the basic integral of quasi-transverse vibrations which satisfies the membrane boundary conditions (and which, in general, leads to a disparity with the non-membrane boundary conditions) and an auxiliary integral of the quasi-transverse vibrations which removes this disparity.

In such an approximate solution, there will be a 'secondary' disparity in the membrane boundary conditions. It can however be removed by constructing the following approximate basic integral of the quasi-transverse vibrations, etc. Thus, the iterative processes in section 9 can be constructed so, as to take the boundary conditions into account.

The conditions of applicability of the decomposition method can be formulated as follows:

10.1 the approximate equations (4.2) of the basic integrals for the quasi-transverse vibrations should have for definite values of λ, non-trivial solutions, satisfying the given membrane boundary conditions;

10.2 there should exist integrals of the dynamic equations of the theory of shells (1.1) to (1.6), whose approximate values are determined by the boundary conditions 10.1;

10.3 auxiliary integrals of the quasi-transverse vibrations should exist, correspondding to the solution of the boundary-value problem 10.1;

10.4 the iterative process of imposing the boundary conditions described at the beginning of this section, should converge. The question of satisfying conditions 10.1 will be discussed in sections 11 and 12. It follows from 10.2 that p and q should satisfy the inequalities in the Table and possibly, also auxiliary conditions (see the end of section 9).

The inequalities in the Table are also necessary for satisfying the condition 10.3. Moreover, it is also necessary that the equality $\lambda = 1/R_2'^2$ is not satisfied at any point on the boundary.

The study of condition 10.4 in the rigorous formulation is as difficult as the study of the existence of the basic and auxiliary integrals. To be logically consistent, we must replace this condition by a formal requirement, that the 'secondary' discrepancy in the membrane boundary conditions diminishes without limit together with h_*. Then the question becomes basically simple. Here one can use the method described in [2] for the analogous static problem. Generally speaking, condition 10.4 will be formally fulfilled whenever conditions 10.1 to 10.3 are satisfied. However, one can encounter exceptions connected with the peculiarities of the theorems on the existence of the solution of dynamic boundary-value problems.

11. Now we turn to condition 10.1, i.e. we will consider the boundary-value problem consisting in the integration of equations (4.2) subject to the membrane boundary conditions. As earlier, we will discuss the oscillations with positive index of variation. Thus, in (4.2) one can retain only the leading derivatives of the unknowns, i.e., the quantities A, B, R_1', R_2', and R_{12} are to be regarded as constants. In resorting to the symbolic method, this allows us to reduce system the (4.2) to the equation

$$LL(\Phi) - \lambda NN(\Phi) = 0$$

$$L = \frac{1}{A^2 R_2'} \frac{\partial^2}{\partial \alpha^2} + \frac{2}{AB} \frac{1}{R_{12}} \frac{\partial^2}{\partial \alpha \, \partial \beta} + \frac{1}{B^2} \frac{1}{R_1'} \frac{\partial^2}{\partial \beta^2}, \quad N = \frac{1}{A^2} \frac{\partial^2}{\partial \alpha^2} + \frac{1}{B^2} \frac{\partial^2}{\partial \beta^2} \tag{11.1}$$

where Φ is a potential function in terms of derivatives of which, all the unknowns are expressed.

The question of the existence of non-trivial solutions of the boundary-value problem for equation (11.1) with homogeneous linear boundary conditions is debatable. Its peculiarity lies in the fact that one must determine eigenvalues of the factor λ preceding the operator NN which is of the same order as the operator LL which does not contain λ. Such problems are hardly ever encountered in the literature, and the associated spectrum has its own peculiar properties which will be demonstrated by some examples.

Example 11.1 Let (1.1) have the form

$$\left(\frac{1}{R^2}-\lambda\right)\frac{\partial^4\Phi}{\partial\alpha^4}-2\lambda\,\frac{\partial^4\Phi}{\partial\alpha^2\,\partial\beta^2}-\lambda\,\frac{\partial^4\Phi}{\partial\beta^4}=0 \qquad (R=\text{const}) \tag{11.2}$$

i.e. the problem is that of the vibration of a circular, cylindrical shell of radius R.

When the coefficient in the brackets on the left-hand side of the equation (11.2) is negative (11.2) can be treated as the homogeneous equation of the flexure of an anisotropic plate. Then it follows that for the usual boundary conditions it does not possess non-trivial solutions. Consequently, all the eigenvalues of λ, if they exist, must lie on the closed interval $(0, R^{-2})$. It is easy to find a case where the spectrum of λ for the equation (11.2) comprises an infinite set of values with the accumulation point $\lambda = 1/R^2$. Such a spectrum is obtained when (11.2) is integrated in a rectangle with boundary conditions corresponding to a hinged support.

With an equation of type (11.1) one can encounter a spectrum which is dense everywhere. In order to show this, let us consider the following example.

Example 11.2 Consider the equation

$$a_1\frac{\partial^2\Phi}{\partial\alpha^2}+b_1\frac{\partial^2\Phi}{\partial\beta^2}-\lambda\left(a_2\frac{\partial^2\Phi}{\partial\alpha^2}+b_2\frac{\partial^2\Phi}{\partial\beta^2}\right)=0 \tag{11.3}$$

where a_1, a_2, b_1 and b_2 are positive constants, and let us integrate it in the rectangle $(0\leqslant\alpha\leqslant\alpha_0;\ 0\leqslant\beta\leqslant\beta_0)$ with the boundary conditions $\Phi = 0$.

Equation (11.3) can be reduced to the form

$$\frac{\partial^2\Phi}{\partial x^2}-\frac{\partial^2\Phi}{\partial y^2}=0 \qquad \left(\rho=\frac{b_1-\lambda b_2}{a_1-\lambda a_2},\ x=\alpha,\ y=\frac{\beta}{\sqrt{\rho}}\right)$$

when the region of integration becomes

$$\left(0\leqslant x\leqslant\alpha_0,\ 0\leqslant y\leqslant\frac{\beta_0}{\sqrt{\rho}}\right)$$

This is the homogeneous Dirichlet problem for the wave equation which was studied in [3 to 6]. In a rectangle it has a non-trivial solution if and only if the ratio of the sides is a rational number. Consequently

$$\rho=\frac{b_1-\lambda b_2}{a_1-\lambda a_2}=\frac{n^2\beta_0^2}{m^2\alpha_0^2} \qquad (n,\ m\text{ — are integers})$$

From this it is easy to derive a formula for the eigenvalues of λ, from which it follows that they are densely distributed everywhere in the interval $(a_1 / a_2, b_1 / b_2)$.

Note. From this example it follows that an equation of type (11.1) can have a spectrum which is dense everywhere. However, this obviously formal result only means that there is a certain condensation of frequencies, since out of all the eigenvalues of λ of equation (11.1) it is only necessary to retain those corresponding to the indices of variation restricted by the inequalities in the Table.

When $a_1 / a_2 = b_1 / b_2$, the spectrum of problem 11.2 degenerates into a point and the eigenfunction of Φ becomes indeterminate. In problems of the theory of shells, such a case will occur when in equation (11.1)

$$LL = \frac{1}{R^2} N \qquad (R = \text{const.})$$

i.e. when the median surface of the shell is a sphere. In discussing this result (the uniqueness of the frequency and the arbitrariness of the form of oscillation), it is necessary to point out that equation (11.1) determines only the first approximate solution. It can be assumed that some iterative process exists and that the obtained formal result means that a certain number of frequencies of the spherical shell differs little from each other and that the associated modes of oscillation can be determined not from the initial but from subsequent stages.

12. If one studies the vibration in which the variation in the direction of one of the coordinates, e.g. in the β direction, predominates over the given region a question arises whether in the first approximation of (11.1), the derivatives with respect to α are negligible compared with those with respect to β, and can be neglected. Of course, for this it is necessary that the resulting equation for the first approximation with the membrane boundary conditions should have non-trivial solutions with a variation of the required type. Let us consider some examples.

Example 12.1 The oscillation of a circular cylindrical shell of radius R (β) with predominant variation along the transverse coordinate β. In this case, equation (11.1) can be put in the form

$$\frac{1}{R^2} \frac{\partial^4 \Phi}{\partial \alpha^4} - \lambda \frac{\partial^4 \Phi}{\partial \beta^4} = 0 \tag{12.1}$$

Assuming that the region of integration is a rectangle with sides parallel to the coordinate axes, equation (12.1) can be solved by the method of separation of variables. Setting

$$\Phi = X(\alpha) Y(\beta)$$

we will have

$$X^{\text{IV}} - kX = 0, \qquad Y^{\text{IV}} - \frac{r}{R^2} Y = 0, \qquad \lambda = \frac{k}{r} \tag{12.2}$$

From this it follows that there exist two sequences of eigenvalues $k_1, k_2, \ldots, k_s, \ldots$ and $r_1, r_2, \ldots, r_t, \ldots$ for the parameters k and r. The first sequence increases with increase in the index of variation p in the direction of the α-curves; the second sequence increases with increase in the index of variation q in the direction of the β-curves.

From the last equation in (12.2) it is clear that λ increases without bounds as p increases, and that it decreases without bounds with increasing q. (Formally, one obtains an unbounded spectrum, although, λ is in fact bounded from above also in the present case because of the necessity of satisfying the inequality $p < q$.)

Example 12.2 Vibration of a shell of revolution with predominant variation along the longitudinal coordinate α. In this case, the simplified equation (11.1) has the form

$$\left(\frac{1}{R_2^2} - \lambda\right) \frac{1}{A^4} \frac{\partial^4 \Phi}{\partial \alpha^4} = 0 \tag{12.3}$$

When R_2 = const, i.e. for circular cylindrical shells and for the sphere, this equation has a single eigenvalue $\lambda = 1 / R_2^2$, and the function Φ remains indeterminate. This result was already obtained for the problem of arbitrary vibrations of a spherical shell (section 11). Now we can add that the vibrations of a circular cylindrical shell exibit the same properties, provided the variation in the longitudinal direction predominates.

When $R_2 = R_2(\alpha)$, equation (12.3) has only trivial solutions. In discussing this result it is necessary to note that in passing from (11.1) to (12.3) the term with the fourth derivative with respect to α which was the dominant term on the left-hand side of equation (11.1), was retained. This is legitimate only in the case when the expression $\lambda - 1 / R_2^2$ is nowhere too small in the region under consideration. This will obviously be not true for values of λ bounded by the inequalities

$$\min \frac{1}{R_2^2} < \lambda < \max \frac{1}{R_2^2} \tag{12.4}$$

Thus, the obtained result means only that the oscillations of the type sought (i.e. the variation with respect to α is always sufficiently large and always greater than the variation with respect to β), cannot take place at frequencies outside the limits imposed by (12.4). However, when these inequalities are fulfilled, then not only the passage from (11.1) to (12.3) but also the use of the momentless equation (11.1) becomes illegitimate.

Problems of the present type also include the problem of the axially symmetric oscillation of a circular conical shell studied in [7]. There use was made of the method of asymptotic integration of the ordinary differential equations, and it was discovered that within the interval of integration there is so-called transition point, i.e. a point at which the coefficient of the highest-order derivative of the degenerate (momentless) equation vanishes.

In the more general problems on the eigenvalues of equations (11.1), one can encounter transition curves. This means a violation of condition 10.1 of the applicability of the first variant of the method of decomposition (section 10).

Example 12.3 The vibrations of a shell of revolution with predominant variation along the transverse coordinate β. In this case the simplified equation (11.1) has the form

$$\left(\frac{1}{R_1^2} - \lambda\right) \frac{1}{B^4} \frac{\partial^4 \Phi}{\partial \beta^4} = 0 \tag{12.5}$$

Once again when $R_1 = \text{const}$ (for the sphere) we obtain a single eigenvalue for λ, and when $R_1 = R_1(\alpha)$ there are no eigenvalues for λ. In this case, this is due to the fact that no vibrations exist that satisfy the imposed conditions.

13. In dynamic problems there is also a second variant of the method of decomposition which has no static analogy. It consists in the fact that the solution in the first approximation is composed of the basic integral for the quasi-membrane oscillations, which satisfy the membrane boundary values, and the corresponding auxiliary integral of the quasi-membrane oscillations which removes the discrepancies in the non-membrane boundary conditions.

The conditions for the realization of the second variant of the decomposition are formulated in the manner similar to that of the first variant in section 10, but the basic and auxiliary integrals of the quasi-transverse oscillations are replaced with those for quasi-membrane oscillations. At the same time, the answer to the question as to when these conditions will be fulfilled is much simpler, than that in the case of quasi-membrane oscillations.

The boundary-value problem of condition 10.1 is identical with the boundary-value problem of the plane theory of elasticity. For the usual boundary conditions, it has a non-trivial solution for definite values of λ.

The region of formal existence of the basic and auxiliary integrals for quasi-membrane oscillations can be determined by inequality (3.2). Its upper boundary coincides with the boundaries of applicability of the classical theory of shells.

From (3.2) and (8.1) it follows that

$$\lambda \gg \frac{1}{R_2'^2} \tag{13.1}$$

Therefore, no points for the auxiliary integrals exist, at which the equality $\lambda = 1 / R_2'^2$ is either exactly or approximately satisfied.

It is easy to settle the question of the formal fulfilment of condition 10.4. In the basic integrals for quasi-membrane oscillations, the membrane factors are significantly larger than the corresponding non-membrane ones. Thus, for example, $\xi, \eta \gg \zeta$. For this it follows that the 'primary' discrepancy in the non-membrane boundary conditions will already be small. In the auxiliary integral for quasi-membrane oscillations, on the other hand, the non-membrane factors dominate. In particular ($\zeta \gg \xi, \eta$). Therefore the 'secondary' discrepancy in the membrane boundary conditions will be quite small. However, if one speaks of satisfying condition 10.4 in the stricter sense, a special case may occur connected with the fact that the auxiliary integrals for quasi-membrane oscillations obviously have a purely oscillatory part, which follows from (13.1) and (7.4).

14. The basic integrals for the quasi-transverse oscillations with large variation

contain sufficient arbitrariness to satisfy all four boundary conditions of the theory of shells. In particular, if these conditions are decomposed into membrane and non-membrane parts, then the non-membrane boundary conditions must be taken into account when integrating equations (6.1) while the membrane boundary conditions, — when integrating equation (6.3). Both boundary-value problems obviously have solutions. The first problem is the boundary-value problem for the flexure of a plate, and the second one is the non-homogeneous static boundary-value problem in the plane theory of elasticity.

15. Now we will formulate the conclusions that follow from the above results. It will be assumed that inequality (3.2) is always satisfied. Free vibrations can be subdivided into quasi-transverse and quasi-membrane types. The quasi-transverse oscillations are characterized by the fact that the relation $\zeta \gg \max(\xi, \eta)$ is satisfied for them. In deriving the first approximation for these oscillations one may neglect the inertial membrane forces. For the quasi-membrane oscillations one has $\max(\xi, \eta) \gg \zeta$ and the inertial membrane forces cannot be neglected.

The quasi-transverse oscillations in their turn are conveniently classified according to their variations. When p and q satisfy both inequalities given for the respective cases in the Table, the variation of the quasi-transverse oscillation will be called average. When at least one of these inequalities is violated, the variation of the quasi-transverse oscillation will be called large.

When certain auxiliary conditions are satisfied (see sections 9 to 12), the quasi-transverse oscillation with average variation can be studied with the aid of the first variant of the method of decomposition of the state of stress (section 10). In this case, the eigenvalues λ, will be determined in the first approximation, from the boundary-value problem consisting of the integration of the dynamic momentless equations (without the inertial shear forces) using the membrane boundary conditions.

The quasi-transverse oscillations with large variation are constructed with the aid of the quasi-transverse integrals. Moreover, the eigenvalues for λ are found in the first approximation from the boundary-value problem consisting of the integration of the equations for the flexural vibration of plates using the non-membrane boundary conditions.

One can also have intermediate quasi-transverse oscillations for which one of the inequalities in the Table becomes an equality.

In the first approximation, such oscillations are studied with the aid of equations which are the dynamic analogs of the equations for a state of stress with large variation. They were used in [8].

Quasi-membrane oscillations with any indices of variation within (3.2) can be constructed with the aid of the second variant of the method of decomposition of the state of stress (section 13). Moreover, the eigenvalues of λ can be determined in the first approximation by integrating the dynamic equations of the plane theory of elasticity using the membrane boundary conditions.

Note. In both variants of the method of decomposition, the auxiliary integrals are needed only in order to make more accurate the states of stress and strain, and have no effect on the first approximation for λ.

The asymptotic estimates of the igenvalues of λ were constructed for all types of vibration. For quasi-transverse oscillations with average variation, these estimates are given in the Table. For quasi-transverse oscillations with large variation and for quasi-membrane oscillations, the estimates can be determined by formulas (6.2) and (5.4), respectively.

With the exception of the quasi-transverse oscillations with average variation, all the different types of oscillation have the usual regulatity, i.e. increase in frequency with increase in variation.

The quasi-transverse oscillations with average variation constitute an exception. Firstly, for them the eigenvalues of λ either remains essentially constant or even decreases with increase in the value of the index of variation of the state of stress. Secondly, for these oscillations (and only for these) the equations determining the first approximation to λ depend on the curvature of the surface. As a consequence, not only the density of the nodal lines but also their configuration (distribution relative to the asymptotic curves) turn out to be important.

From the estimates of λ appearing in the Table, one can draw conclusions about the lowest frequency of oscillation of shells not having positive curvature. In such a shell it is possible to have oscillations with predominantly asymptotic nodal lines, i.e. with a predominance of variation in the direction perpendicular to the asymptotic curves. In this case, when the variation is increased, λ will first decrease (as long as the inequality in the Table is satisfied) and then (when one or other of the inequalities is violated) it will increase. The minimum value of λ will be attained at the transition from decrease to increase when intermediate quasi-transverse oscillations take place. This result was obtained by another method in [8].

When $\rho = q = 0$, the oscillations of the shell cannot be separated into quasi-transverse and quasi-membrane. For such values of p and q the method of decomposition, i.e. the solution is formed from the basic integral determined in the first approximation by the dynamic momentless equations and the auxiliary integral (section 7), can be used in investigations of oscillations. However, in this case one must take, in the momentless equations, all inertial terms into account, and this will radically change the nature of the corresponding boundary-value problem, which will now be of third degree in λ instead of the first. The question of the properties of the solution of such a problem calls for a separate investigation.

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